1st Year: Elec. Eng.(Power) Mathematics 2-B

Answer All Questions Mid-Term Exam (8 – 5 – 2011) Time 1 Hour

[1]Find the series solution of the equation $x^2 y'' - x y' + (1+x)y = 0$

[2]Compute the integrals: (a) $\int_0^{\infty} \frac{e^{-2x}}{x\sqrt{x}}\,dx$ (b) $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx$ (c) $\int_0^{\infty} \frac{2^x - 3^x}{x}\,dx$

[3]Find F(s) to the functions: (a) $f(t) = (t - 2\sin t)^2$ (b) $f(t) = (t-2)\sin(t-2),\ t > 2$

[4] Define the Dirac function $\partial_0(t)$ and show that $L\{\partial_0(t)\} = 1$

[5]Using L.T solve the equation: $y'' - 4y' + 4y = [t\ e^t]^2$, y(0) = y`(0) = 0

Good Luck *Dr. Mohamed Eid*

Model Answer

[1] Since $p(x) = -\frac{1}{x},\ q(x) = \frac{1+x}{x^2}$ are not analytic functions at x = 0.

But $x\,p(x) = -1$ and $x^2 q(x) = 1 + x$ are analytic functions at x = 0.

Then $y(x) = \sum_{n=0}^{\infty} a_n x^{n+c}$, $y' = \sum_{n=0}^{\infty} a_n (n+c) x^{n+c-1}$ and

$$y'' = \sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2}$$

From the given differential equation, we get

$$\sum_{n=0}^{\infty} (n+c)(n+c-1) a_n x^{n+c} - \sum_{n=0}^{\infty} (n+c) a_n x^{n+c} + \sum_{n=0}^{\infty} a_n x^{n+c} + \sum_{n=0}^{\infty} a_n x^{n+c+1} = 0$$

Then $c(c-1) a_0 x^c - c a_0 x^c + a_0 x^c + \sum_{n=1}^{\infty} (n+c)(n+c-1) a_n x^{n+c} - \sum_{n=1}^{\infty} (n+c) a_n x^{n+c} +$

$$+ \sum_{n=1}^{\infty} a_n x^{n+c} + \sum_{n=0}^{\infty} a_n x^{n+c+1} = 0.$$

In the first, second and third sum, put n = m.

In the fourth sum, put n + 1 = m.

Then $(c^2 - 2c + 1) a_0 x^c + \sum_{m=1}^{\infty} \left[((m+c)(m+c-1) - (m+c) + 1) a_m + a_{m-1} \right] x^{m+c} = 0$

Equating the coefficients, we get

The indicial equation is $c^2-2c+1=0$. Then c =1.

The recurrence relation is:

$$((m+c)(m+c-2)+1)\,a_m+a_{m-1}=0,\ m=1,2,3\ldots$$

Then $a_m=\dfrac{-a_{m-1}}{(m+c)(m+c-2)+1},\ m=1,2,3\ldots$

If m = 1, then $a_1=\dfrac{-a_0}{(c+1)(c-1)+1}=\dfrac{-a_0}{c^2}$

If m = 2, then $a_2=\dfrac{-a_1}{(c+2)(c)+1}=\dfrac{a_0}{c^2(c+1)^2}$

If m = 3, then $a_3=\dfrac{-a_2}{(c+3)(c+1)+1}=\dfrac{-a_0}{(c(c+1)(c+2))^2}$

Then y = $x^c a_0\left[1-\dfrac{x}{c^2}+\dfrac{x^2}{(c(c+1))^2}-\dfrac{x^3}{(c(c+1)(c+2))^2}\ldots\right]$

$$\frac{\partial y}{\partial c}=x^c\ln x\,a_0\left[1-\frac{x}{c^2}+\frac{x^2}{(c(c+1))^2}-\frac{x^3}{(c(c+1)(c+2))^2}\ldots\right]$$

$$+a_0x^c\left[0+\frac{2x}{c^3}-\frac{2(2c+1)x^2}{(c(c+1))^3}+\frac{2(3c^2+6c+2)x^3}{(c(c+1)(c+2))^3}\ldots\right]$$

Putting c = 1, then u(x) = $a_0x\left[1-\dfrac{x}{1^2}+\dfrac{x^2}{(1.2)^2}-\dfrac{x^3}{(1.2.3)^2}\ldots\right]$

and v(x) = $a_0x\ln x\left[1-\dfrac{x}{1^2}+\dfrac{x^2}{(1.2)^2}-\dfrac{x^3}{(1.2.3)^2}\ldots\right]+a_0x\left[\dfrac{2x}{1^3}-\dfrac{6x^2}{(1.2)^3}+\dfrac{22x^3}{(1.2.3)^3}\ldots\right]$.

Then y(x) =A u(x) + B v(x).

[2](a) Put y = 2x, we get $\int_0^\infty \frac{e^{-2x}}{x\sqrt{x}}\,dx = \sqrt{2}\int_0^\infty y^{-3/2}e^{-y}dy = \sqrt{2}\Gamma(-1/2) = -2\sqrt{2\pi}$

(b)Since $\frac{x^2}{1+x^4}$ is even function. Then $\int_{-\infty}^\infty \frac{x^2}{1+x^4}\,dx = 2\int_0^\infty \frac{x^2}{1+x^4}\,dx$

Put $y = x^4$, we get $\int_{-\infty}^\infty \frac{x^2}{1+x^4}\,dx = 2.\frac{1}{4}\int_0^\infty \frac{y^{-1/4}}{1+y}dy = \frac{1}{2}B(\frac{3}{4},\frac{1}{4}) = \frac{\sqrt{2}}{2}\pi$

(c) Since $L\{2^x - 3^x\} = \frac{1}{s-\ln 2} - \frac{1}{s-\ln 3}$

and $L\{\frac{2^x-3^x}{x}\} = \int_s^\infty (\frac{1}{s-\ln 2} - \frac{1}{s-\ln 3})ds = \ln\frac{s-\ln 3}{s-\ln 2} = \int_0^\infty \frac{2^x-3^x}{x}e^{-sx}dx$

Putting s = 0, we get $\int_0^\infty \frac{2^x-3^x}{x}\,dx = \ln\frac{\ln 3}{\ln 2}$

[3](a)Since $f(t) = (t-2\sin t)^2 = t^2 - 4t\sin t + 4\sin^2 t = t^2 - 4t\sin t + 2 - 2\cos 2t$

Then $F(s) = \frac{2}{s^3} + 4[\frac{1}{1+s^2}]' + \frac{2}{s} - \frac{2s}{4+s^2} = \frac{2}{s^3} + \frac{-8s}{(1+s^2)^2} + \frac{2}{s} - \frac{2s}{4+s^2}$

(b) Let $g(t) = t\sin t$ and $L\{g(t)\} = -[\frac{1}{1+s^2}]' = \frac{2s}{(1+s^2)^2}$

Then $L\{f(t)\} = F(s) = L\{g(t-2)\} = L\{(t-2)\sin(t-2)\} = \frac{2s}{(1+s^2)^2}e^{-2s}$

[4] Dirac Delta Function

Let $F_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, t \in [0,\varepsilon] \\ 0, t \in (\varepsilon,\infty) \end{cases}$

The Dirac delta function is defined by: $\partial_0(t) = \lim_{\varepsilon\to 0} F_\varepsilon(t)$

The Laplace transform of $\partial_0(t)$ is 1.

Proof

$$L\left\{\partial_0(t)\right\} = \underset{\varepsilon\to 0}{\mathrm{Lim}}\int_0^\infty F_\varepsilon(t)e^{-st}dt = \underset{\varepsilon\to 0}{\mathrm{Lim}}\int_0^\varepsilon \frac{1}{\varepsilon}e^{-st}dt = \underset{\varepsilon\to 0}{\mathrm{Lim}}\left[\frac{e^{-st}}{-s\varepsilon}\right]_0^\varepsilon = \underset{\varepsilon\to 0}{\mathrm{Lim}}\frac{1-e^{-s\varepsilon}}{-s\varepsilon} = \underset{\varepsilon\to 0}{\mathrm{Lim}}\frac{se^{-s\varepsilon}}{s} = 1$$

[5] Since $y`` - 4y` + 4y = [t\ e^t]^2 = t^2e^{2t}$. Then $L\{y``\} - 4L\{y`\} + 4L\{y\} = L\{t^2e^{2t}\}$

Then $[s^2Y(s) - sy(0) - y`(0)] - 4[sY(s) - y(0)] + 4Y(s) = \dfrac{2}{(s-2)^3}$

From the initial conditions, we get $s^2Y(s) - 4sY(s) + 4Y(s) = \dfrac{2}{(s-2)^3}$

Then $Y(s) = \dfrac{2}{(s-2)^5}$. Then, the solution is $y(t) = \dfrac{2}{4!}t^4e^{2t}$

Benha University Faculty of Eng.- Shoubra Eng. Math. & Phy. Department		1st Year: Elec.(Power) Mathematics 2-B Date: 14 / 6 / 2011

الزمن 3 ساعات **(تخلفات)** الامتحان (5) أسئلة فى صفحة واحدة و المطلوب إجابة كل الأسئلة | Marks

[1] Find the series solution of the equations: 20

(a) $y'' + xy = 0$ (b) $x^2y'' + xy' - y = 0$

[2] Evaluate the integrals: 20

(a) $\int_0^\infty \frac{e^{-2x}}{x^{3/2}}\,dx$ (b) $\int_0^{\pi/2} \sqrt{\tan x}\,dx$ (c) $\int_0^1 x.P_3(x)dx$ (d) $\int_0^\infty \frac{3^t - 2^t}{t}\,dt$

[3](a)Prove that: If f(t) is function with Laplace transformation F(s). Then

$$L\{f(t)/t\} = \int_s^\infty F(s)ds$$ 10

(b)Find the Laplace transformation of f(t) = $(e^{3t} - 2t)^2$

(c)Find the inverse Laplace transform of $F(s) = \frac{1}{s^3(s^2+1)}$ 5

[4] Using Laplace transformations, solve the equations:

(a) $y'' - 4y' = t$, $y(0) = 0$, $y'(0) = 3$ 5

(b) $y'' + 4y' - 4y = [te^t]^2$, $y(0) = 0$, $y'(0) = 0$

(c) $y'' + y = \sin t$, $y(0) = 0$, $y'(0) = 1$

6

[5] Solve the P.D. equations:

(a) $3u_x + 4u_y + 25u = 20$ (b) $u_x - u_y + u = 0$, $u(0,y) = e^{2y}$ 6

(c) $u_{xx} - 4u_{xy} + 3u_{yy} = \sin(2x + 3y)$

8

8 + 6

6

Good Luck *Dr. Mohamed Eid*

Benha University
Faculty of Eng.- Shoubra
Eng. Math. & Phy. Department

1^{st} Year: Surv. Eng.
Mathematics B
Date: 12 / 6 / 2011

الزمن 3 ساعات **(تخلفات)** الامتحان (5) أسئلة فى صفحة واحدة و المطلوب إجابة كل الأسئلة

	Marks
[1](a)Find the line y = ax + b that fits the data (1, 2), (2, 4), (4, 5), (5, 3), (6, 10).	10
(b)Write the table of differences of the data (1, 1), (2, 4), (3, 12), (4, 15), (5, 20) and then find the value of y at x = 1.5	10
[2](a)Find the logarithmic curve y = a ln x + b that fits the data: (1, 3), (3, 4), (4, 6), (5, 12), (7, 20).	10
(b)Find the value of x at y = 3 from the data: (1, 2), (3, 4), (5, 9), (7, 11).	10
[3]Find the following integrals:	
(a) $\int_0^2 \int_0^x (xy^2)dydx$ (b) $\int_0^3 \int_0^{\sqrt{9-x^2}} \frac{1}{\sqrt{x^2+y^2}}dydx$	6+6
(c) $\int_{(0,0)}^{(2,4)} (x^2+2y)dx+(x-y)dy$ through $y = x^2$	8
[4](a)Find $B=\sqrt{A}$ where $A=\begin{bmatrix}1 & 2\\ 0 & 9\end{bmatrix}$.	8
(b)Write the Hessain matrix of the function $f(x)=x^4+2^y+xy\sin z$	6
(c)Show that $P = 5x^2+3y^2+4z^2-2xy-4xz$ is positive definite.	6
[5](a)Write the Fourier integral of the function $f(x)=\begin{cases}x, & \lvert x\rvert \le 2\\ 0, & \lvert x\rvert > 2\end{cases}$	8
(b)Write the Fourier series of the function $f(x)=x,\ x\in[-\pi,\pi],\ f(x+2\pi)=f(x)$ Also, find the sum $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$	12

Good Luck *Dr. Mohamed Eid*

Benha University Faculty of Eng.- Shoubra Eng. Math. & Phy. Department		1st Year: Elec. Eng.(Power) Mathematics 2-B Date: 3 / 7 / 2011

Time 3 Hours	الامتحان (5) أسئلة فى صفحة واحدة و المطلوب إجابة كل الأسئلة	Marks
[1] Find the following integrals: (a) $\int_0^\infty \frac{1}{\sqrt{xe^x}}\,dx$ (b) $\int_0^2 \frac{y^2}{\sqrt{2-y}}dy$ (c) $\int_0^{\pi/2} \sqrt{\cot z}\;dz$ (d) $\int_0^\infty \frac{2\sin 3t.\sin 4t}{t}\;dt$		20
[2](a) Find the series solution of the equation: $y``-xy=2x$		8
(b) Using Laplace transforms, solve the equation: $y``-3y`+2y=e^{2t}$, $y(0)=y`(0)=0$		8
[3](a)Find the Laplace transformation of the functions: (i)$f(t)=(e^{-t}-2t)^2$ (ii) $f(t)=\sqrt{t}+e^{3t}\sin t$		10
(b)Find the inverse Laplace transform of : (i) $F(s)=\frac{1}{s^2(s-1)}$ (ii) $F(s)=\frac{s}{s^2-3s+2}$		10
[4] Solve the following partial differential equations: (a) $u_x-2u_y+3u=0$, $u(0,y)=e^{3y}$ (b) $3u_x+4u_y=5(x^2+y^2)$ (c) $u_{xx}-3u_{xy}=e^{2x+y}$ (d) $u_{xx}-3u_{xy}+2u_{yy}=\cos(x+y)$		24
[5](a) Prove that: $B(m,n)=\frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$		10
(b)Solve the linear system: $\begin{bmatrix} x` \\ y` \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$		10

Good Luck *Dr. Mohamed Eid*

Model Answer

[1] (a) $\int_0^\infty \frac{1}{\sqrt{xe^x}}\, dx = \int_0^\infty x^{-\frac{1}{2}}\, e^{-\frac{1}{2}x}\, dx$. Put $x = 2y$, $dx = 2dy$

Then $I = \sqrt{2}\int_0^\infty y^{-\frac{1}{2}}\, e^{-y}\, dy = \sqrt{2}\Gamma(1/2) = \sqrt{2\pi}$

(b) $\int_0^2 \frac{y^2}{\sqrt{2-y}} dy$. Put $y = 2x$, $dy = 2dx$

Then $I = \frac{8}{\sqrt{2}}\int_0^\infty x^2\, (1\text{-}x)^{-\frac{1}{2}}\, dx = 4\sqrt{2}\ B(3,\frac{1}{2}) = \frac{64\sqrt{2}}{15}$

(c) $\int_0^{\pi/2} \sqrt{\cot z}\ dz = \int_0^{\pi/2} (\cos z)^{\frac{1}{2}}.(\sin z)^{-\frac{1}{2}}\ dz = \frac{1}{2}B(\frac{1}{4},\frac{3}{4}) = \frac{\pi}{\sqrt{2}}$

(d) $\int_0^\infty \frac{2\sin 3t.\sin 4t}{t}\ dt$

Since $2\sin 3t\cos 4t = \cos t - \cos 7t$ and $L\{\cos t - \cos 7t\} = \frac{s}{s^2+1} - \frac{s}{s^2+49}$

Then $L\{\frac{\cos t - \cos 7t}{t}\} = \int_s^\infty (\frac{s}{s^2+1} - \frac{s}{s^2+49})ds = \frac{1}{2}\ln\frac{s^2+49}{s^2+1} = \int_0^\infty (\frac{\cos t - \cos 7t}{t})e^{-st}dt$

Putting $s = 0$, then $I = \frac{1}{2}\ln 49 = \ln 7$

[2](a) From the equation: $y`` - xy = 2x$

Since $p(x) = 0$ and $q(x) = -x$ are analytic functions at $x = 0$.

Then the power series solution takes the form: $y = \sum_{n=0}^{\infty} a_n x^n$

Then $y` = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y`` = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substituting in the given equation, we get $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 2x$

Then $2 a_2 x^0 + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 2x$

In the first sum, put n – 2 = m

In the second sum, put n + 1 = m

Then $2 a_2 + \sum_{m=1}^{\infty} \left[(m+2)(m+1) a_{m+2} - a_{m-1}\right] x^m = 2x$

Equating the coefficients in both sides, we get

$2a_2 = 0$, then $a_2 = 0$

When $m = 1$: $6a_3 - a_0 = 2$ <u>Coefficient x</u>

$\left[(m+2)(m+1) a_{m+2} - a_{m-1}\right] = 0$, $m = 2, 3, 4, \ldots$

Thus the recurrence relation (R.R) is:

$$a_{m+2} = \frac{a_{m-1}}{(m+1)(m+2)}, \quad m = 2, 3\ldots$$

If $m = 2$, then $a_4 = \frac{a_1}{12}$

If $m = 3$, then $a_5 = \frac{a_2}{20} = 0$

If $m = 4$, then $a_6 = \frac{a_3}{30} = \frac{2 + a_0}{180}$

Then $y(x) = a_0 + a_1 x + a_2 x^2 \ldots$

$$= a_0 + a_1 x + 0 + \frac{2+a_0}{6}x^3 + \frac{a_1}{12}x^4 + 0 + \frac{2+a_0}{180}x^6 + \ldots$$

$$= a_0[1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \ldots] + a_1[x + \frac{1}{12}x^4 + \ldots] + [\frac{1}{3}x^3 + \frac{1}{90}x^6 + \ldots]$$

(b) Since $L\{ y`` - 3y` + 2y\} = L\{ e^{2t} \}$

Then $(s^2 Y - sy(0) - y`(0)) - 3(sY - y(0)) + 2Y = \frac{1}{s-2}$

From the conditions $y(0) = y`(0) = 0$

Then, we get $(s^2 - 3s + 2)Y = \frac{1}{s-2}$ Or $Y = \frac{1}{(s-1)(s-2)^2}$

Using methods of partial fractions, we get $Y = \frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{(s-2)^2}$

Then, the solution of the equation $y(t) = e^t - e^{2t} + te^{2t}$

[3](a)(i) Since $f(t) = (e^{-t} - 2t)^2 = e^{-2t} + 4t^2 - 4te^{-t}$. Then $F(s) = \frac{1}{s+2} + \frac{8}{s^3} - \frac{4}{(s+1)^2}$

(ii)Since $f(t) = \sqrt{t} + e^{3t}\sin t$. Then $F(s) = \frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{(s-3)^2 + 1}$

(b)(i) $F(s) = \frac{1}{s^2(s-1)}$. Since $L^{-1}\{\frac{1}{s-1}\} = e^t$ and $L^{-1}\{\frac{1}{s(s-1)}\} = \int_0^t e^t dt = e^t - 1$

Then $f(t) = L^{-1}\{\frac{1}{s^2(s-1)}\} = \int_0^t (e^t - 1)dt = e^t - t$

(ii) Using methods of partial fractions, we get $F(s)=\dfrac{s}{s^2-3s+2}=\dfrac{2}{s-2}-\dfrac{1}{s-1}$

Then $f(t)=2e^{2t}-e^{t}$

[4](a) $u_x-2u_y+3u=0$, $u(0,y)=e^{3y}$

The required solution takes the form $u(x,y)=e^{ax+by}$. Then $u_x=au$, $u_y=bu$.

Substitute in the given equation, we get (a – 2b + 3)u = 0.

Then a – 2b + 3 = 0 and a = 2b – 3. Then $u(x,y)=e^{(2b-3)x+by}$

From the given condition, $u(0,y)=e^{3y}=e^{by}$. Then a = 3 = b.

Then the required solution is $u(x,y)=e^{3x+3y}$.

(b) $3u_x+4u_y=5(x^2+y^2)$

Let α = x cosθ + y sinθ and β = – x sinθ + y cosθ.

Then $u_x=u_\alpha\alpha_x+u_\beta\beta_x=u_\alpha\cos\theta-u_\beta\sin\theta$

$u_y=u_\alpha\alpha_y+u_\beta\beta_y=u_\alpha\sin\theta+u_\beta\cos\theta$

Substituting in the given equation, we get

$$3(\cos\theta.u_\alpha-\sin\theta.u_\beta)+4(\sin\theta.u_\alpha+\cos\theta.u_\beta)=5(\alpha^2+\beta^2)$$

Since u(x , y) = w(α , β). Then

$$\left[3\cos\theta+4\sin\theta\right]w_\alpha+\left[-3\sin\theta+4\cos\theta\right]w_\beta=5(\alpha^2+\beta^2)$$

If the coefficient of w_β is zero, that is, – 3sinθ + 4cosθ = 0.

Then $\tan\theta = \frac{4}{3}$, $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$.

Then, we get $w_\alpha = \alpha^2 + \beta^2$

$$w = \int (\alpha^2 + \beta^2) d\alpha = \frac{1}{3}\alpha^3 + \alpha\beta^2 + c(\beta).$$

Then $u(x,y) = \frac{1}{3}\left(\frac{3x+4y}{5}\right)^3 + \frac{3x+4y}{5}\left(\frac{-4x+3y}{5}\right)^2 + c\left(\frac{-4x+3y}{5}\right)$

where $c\left(\frac{-4x+3y}{5}\right)$ is arbitrary function.

(c) $u_{xx} - 3u_{xy} = e^{2x+y}$. Since the C.E. is $k^2 - 3k = 0$. Then $k = 0$, $k = 3$.

Then $u_c = f_1(y + 0x) + f_2(y + 3x)$

$$u_I = \frac{1}{D^2 - 3DE} e^{2x+y} = \frac{1}{4-6} e^{2x+y} = \frac{1}{-2} e^{2x+y}$$

The general solution is $u(x, y) = u_c + u_I$

(d) $u_{xx} - 3u_{xy} + 2u_{yy} = \cos(x + y)$. Since the C.E. is $k^2 - 3k + 2 = 0$. Then $k = 1$, $k = 2$.

Then $u_c = f_1(y + x) + f_2(y + 2x)$

$$u_I = \frac{1}{D^2 - 3DE + 2E^2}\cos(x+y) = \frac{1}{-1+3-2}\cos(x+y) = \frac{1}{(D-E)(D-2E)}\cos(x+y)$$

Assume that $y + x = c_1$ and $y + 2x = c_2$. Then

$$\frac{1}{(D-E)}\cos(x+y) = \int \cos(x + c_1 - x)dx = \int \cos c_1 dx = x\cos c_1 = x\cos(x+y)$$

$$\frac{1}{D-2E} x\cos(x+y) = \int x\cos(x+c_2-2x)dx$$

$$= \int x\cos(c_2-x)dx \qquad \text{(Integrate by parts)}$$

$$= x\sin(c_2-x)+\cos(c_2-x)$$

$$= x\sin(y+x)+\cos(y+x)$$

Then $u_I = x\sin(y+x)+\cos(y+x)$

The general solution is $u(x, y) = u_c + u_I$

[5](a) Theorem: $B(m,n) = \dfrac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$

(b) The linear system: $\begin{bmatrix} x` \\ y` \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$

The coefficient matrix: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Then $|A - mI| = \begin{vmatrix} 1-m & 2 \\ 2 & 1-m \end{vmatrix} = (1-m)^2 - 4 = m^2 - 2m - 3 = 0$. Then $m = 3, \ -1$.

The characteristic value problem is:

$$\begin{bmatrix} 1-m & 2 \\ 2 & 1-m \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $m = 3$, then we get the linear system: $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We get: $2a - 2b = 0$. Putting $b = 1$, we get $a = 1$.

Then, the eignvector is $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$

If $m = -1$, then we get the linear system: $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We get: $2a + 2b = 0$. Putting $b = 1$, we get $a = -1$.

Then, the eignvector is $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$

The fundamental matrix is $X = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}$

Then $|X| = 2e^{2t}$

$$X^{-1} = \frac{1}{2e^{2t}}\begin{bmatrix} e^{-t} & e^{-t} \\ -e^{3t} & e^{3t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-3t} & e^{-3t} \\ -e^{t} & e^{t} \end{bmatrix}$$

$$X^{-1}f(t) = \frac{1}{2}\begin{bmatrix} e^{-3t} & e^{-3t} \\ -e^{t} & e^{t} \end{bmatrix}\begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-2t} + e^{-4t} \\ -e^{2t} + 1 \end{bmatrix}$$

$$\int (X^{-1}f(t))dt = \frac{1}{2}\begin{bmatrix} \frac{-1}{2}e^{-2t} - \frac{1}{4}e^{-4t} \\ -\frac{1}{2}e^{2t} + t \end{bmatrix}$$

$$v(t) = X\int (X^{-1}f(t))dt = \frac{1}{2}\begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}\begin{bmatrix} \frac{-1}{2}e^{-2t} - \frac{1}{4}e^{-4t} \\ -\frac{1}{2}e^{2t} + t \end{bmatrix} = \frac{1}{2}\begin{bmatrix} (-t - \frac{1}{4})e^{-t} \\ (t - \frac{1}{4})e^{-t} - e^{t} \end{bmatrix}$$

The general solution is $\begin{bmatrix} x \\ y \end{bmatrix} = c_1X_1 + c_2X_2 + v(t)$